Study of the effect of parameters on the decay rate of a fourth order problem

Control of Partial Differential Equations in Hauts-De-France 2023

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The equation

We study the fourth-order evolution equation:

$$\partial_{tt} u(x,t) + a\partial_{xxxx} u(x,t) + b\partial_t u(x,t) + \alpha\partial_t u(\xi,t)\delta_\xi = 0,$$

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- u: transverse displacement of the bridge deck (identified with $[0,1]$);
- δ_ξ: presence of a shape memory alloy cable at $x = \xi$.

Boundary and initial conditions

We couple the equations with boundary conditions

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and initial conditions

$$u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x), \qquad x \in (0,1).$$

Energy and dissipation

We define the energy of a solution $u(x,t)$ by

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Hence, the system is dissipative, in the sense that the energy decreases.

Functional setting

To use the semigroup theory, we want to rewrite the problem as

$$\begin{cases} \partial_t U = \mathcal{A}_{\alpha} U, \\ U(0) = U_0, \end{cases}$$

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Remark

The norm $U \mapsto (U, U)_{\mathcal{H}}^{1/2}$ is equivalent to the usual norm of $\mathcal{H}$.

Well-posedness of the Cauchy problem

Using the Lumer-Phillips Theorem, we can show the following

Theorem (Existence and uniqueness)

(1) *If* $U_0 \in \operatorname{Dom} \mathcal{A}_\alpha$, *then the Cauchy problem has a unique strong solution*

$$U \in C^0([0,+\infty[, \operatorname{Dom} \mathcal{A}_\alpha) \cap C^1([0,+\infty[, \mathcal{H}).$$

(2) *If* $U_0 \in \mathcal{H}$, *then the Cauchy problem has a unique weak solution*

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The damping rate

Theorem (Régnier (2022))

The system of eigenvectors of $\mathcal{A}_\alpha$ constitutes a Riesz basis in $\mathcal{H}$.

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Corollary (Régnier (2022))

The optimal energy decay rate of the equation, i.e. the smallest $\omega(\alpha) < 0$ such that there exists $C > 0$ with

$$E(t) \leq Ce^{2\omega(\alpha)t}\|U_0\|^2_{\mathrm{Dom}\,\mathcal{A}_\alpha}$$

for all $t \geq 0$ is given by

$$\omega(\alpha) := \sup\left\{\Re(\mu) \mid \mu \textit{ is an eigenvalue of } \mathcal{A}_\alpha\right\}.$$

The role of the parameter α

A natural question

How does the decay rate $\omega(\alpha)$ depend on α?

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An important assumption

To avoid any "resonance phenomena", we will assume that $\xi \notin \mathbb{Q}$.

The characteristic equation

Proposition

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$$\begin{aligned}&(\mu + b)\sinh(\lambda)\sin(\lambda)\\ &+ \alpha\lambda\Big[\sin(\lambda)\sinh(\lambda\xi)\sinh(\lambda(1-\xi)) - \sinh(\lambda)\sin(\lambda\xi)\sin(\lambda(1-\xi))\Big] = 0,\end{aligned}$$

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Remark

Replacing λ by $i\lambda$, $-\lambda$ or $-i\lambda$ leads to an equivalent equation.

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Finding the eigenvalues of $\mathcal{A}_\alpha$ amounts to find roots of the function

$$F(\mu;\alpha) := 2\mu(\mu+b)F_0(\lambda) + \alpha\mu\lambda F_1(\lambda),$$

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- $\alpha \to +\infty \rightsquigarrow$ *roots of* F_1*.*

Dependence of the roots on parameters

A general fact from complex analysis

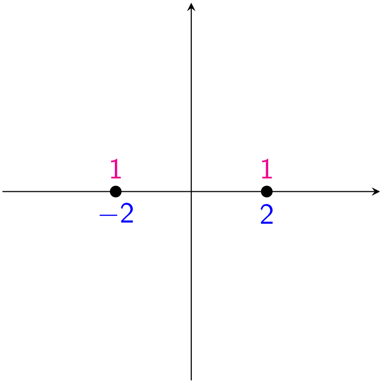
Theorem ("Holomorphic implicit function Theorem", very roughly stated)

Roots of holomorphic functions depend ***continuously, including multiplicities,*** *on the parameters, and the branches of roots are holomorphic.*

Dependence of the roots on parameters

A simple example: roots of $z \mapsto z^2 + c$

$$z \mapsto z^2 - 4$$



Blue: values. Red: multiplicities.

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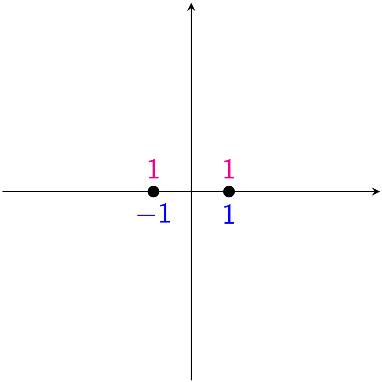
$$z \mapsto z^2 - 1$$



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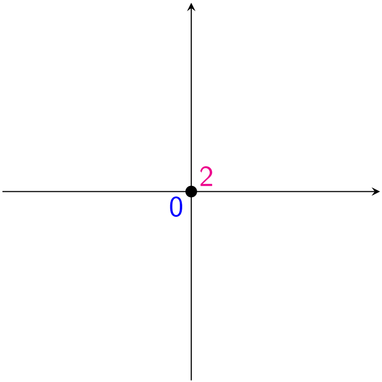

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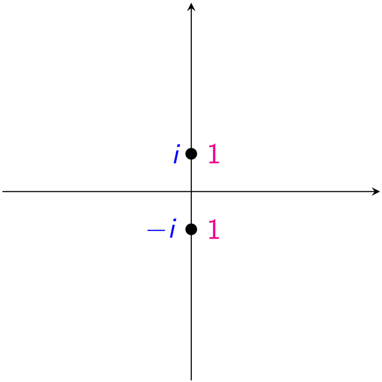

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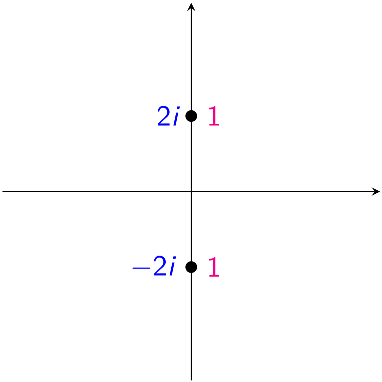

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The case $\alpha = 0$: roots of $\lambda \mapsto F_0(\lambda)$

A computation

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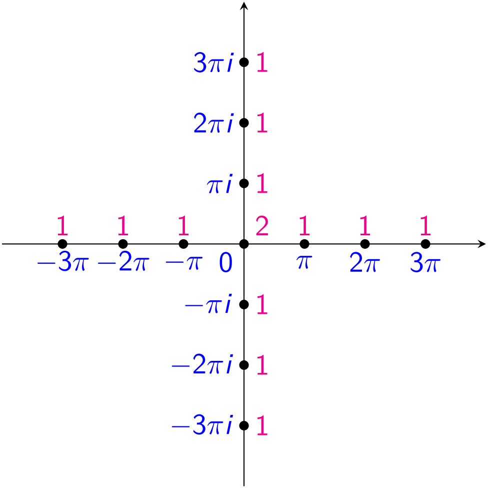
Graphical representation in the λ plane



The case $\alpha = 0$: roots of $\mu \mapsto F_0(\lambda(\mu))$

Graphical representation in the μ plane ($a = 0.05, b = 3$)

$$\lambda(\mu) = \sqrt[4]{-\frac{b\mu + \mu^2}{a}}$$

i.e.

$$\mu^2 + b\mu + a\lambda^4 = 0$$

so that

$$\mu(\lambda) = \frac{-b \pm \sqrt{b^2 - 4a\lambda^4}}{2}$$

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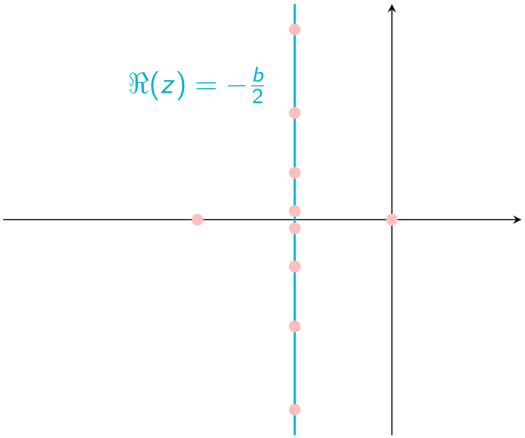

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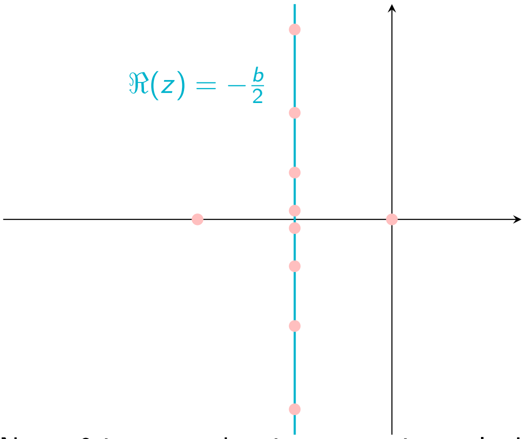

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Note: 0 is a root, but is *not* an eigenvalue!

Roots of $\lambda \mapsto F_1(\lambda)$

The strategy: a continuation argument

We write

$$F_1(\lambda) = s(\lambda) - t(\lambda)$$

where

$$\begin{aligned} s(\lambda) &:= \sin(\lambda)\sinh(\lambda\xi)\sinh(\lambda(1-\xi)) \\ t(\lambda) &:= \sinh(\lambda)\sin(\lambda\xi)\sin(\lambda(1-\xi)). \end{aligned}$$

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Strategy: study roots of

$$\tilde{F}_\gamma(\lambda) := s(\lambda) - \gamma t(\lambda).$$

as γ varies from 0 to 1.

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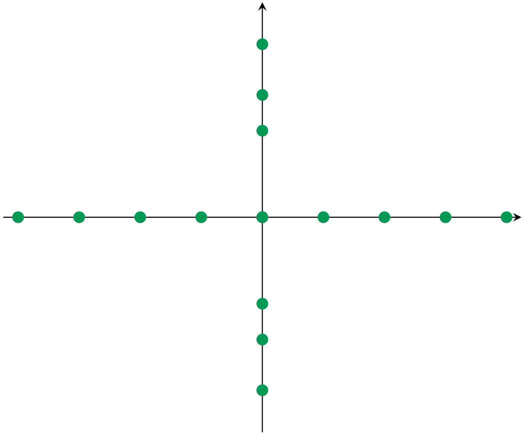
All those roots have multiplicity one, except 0.

Roots of $\lambda \mapsto F_1(\lambda)$

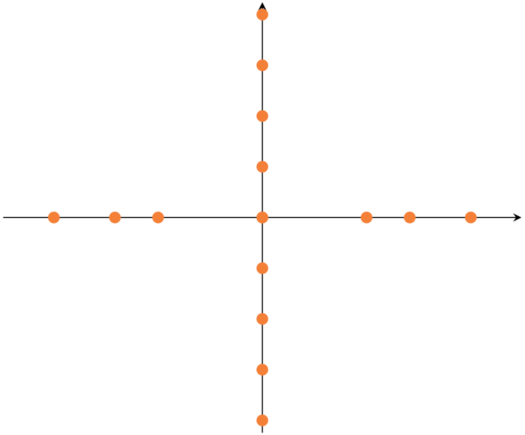
Roots of s and t: graphical representation in the λ plane



Green: roots of s

Roots of $\lambda \mapsto F_1(\lambda)$

Roots of s and t: graphical representation in the λ plane



Orange: roots of t

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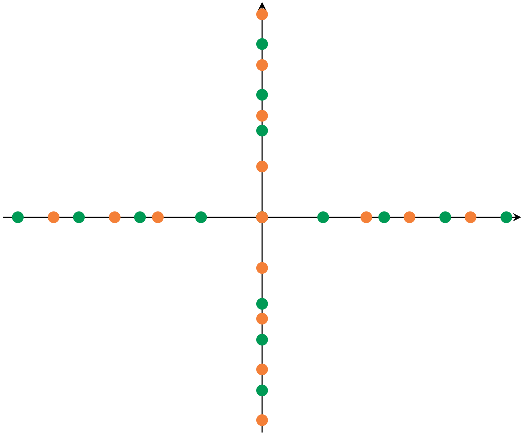

Green: roots of s

Orange: roots of t

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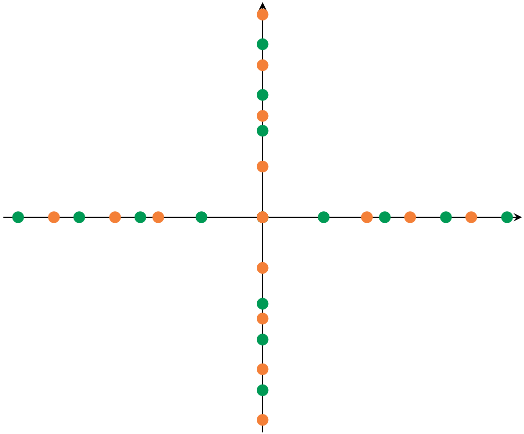

Green: roots of s

Orange: roots of t

Can you see something?

Roots of $\lambda \mapsto F_1(\lambda)$

A detour through number theory: Beatty's Theorem

Theorem (Rayleigh (1894) - Beatty (1927))

Let $0 < r < 1$ *be irrational. Define the sets*

$$A := \left\{ \left\lfloor \frac{n}{r} \right\rfloor \;\middle|\; n \in \mathbb{Z}^{>0} \right\}, \qquad B := \left\{ \left\lfloor \frac{n}{1-r} \right\rfloor \;\middle|\; n \in \mathbb{Z}^{>0} \right\}.$$

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S. Beatty "Problem 3173". American Mathematical Monthly. 33 (3): p. 159 (1926).

Roots of $\lambda \mapsto F_1(\lambda)$

Beatty's Theorem: a numerical example

Let us take $r = \sqrt{2} - 1$. Then (using a little script),

$$A = \left\{ \left\lfloor \frac{n}{r} \right\rfloor \;\middle|\; n \in \mathbb{Z}^{>0} \right\}$$
$$= \left\{ 2, 4, 7, 9, 12, 14, 16, 19, \ldots \right\}$$

and

$$B = \left\{ \left\lfloor \frac{n}{1-r} \right\rfloor \;\middle|\; n \in \mathbb{Z}^{>0} \right\}$$
$$= \left\{ 1, 3, 5, 6, 8, 10, 11, 13, 15, 17, 18, 20, \ldots \right\}.$$

Roots of $\lambda \mapsto F_1(\lambda)$

The continuation argument for $\tilde{F}_\gamma$: main ideas

Main ideas:

Roots of $\lambda \mapsto F_1(\lambda)$

The continuation argument for $\tilde{F}_\gamma$: main ideas

Main ideas:

- Using Beatty's Theorem, we can *prove* that the roots of s and t are intertwined as we saw. This means that we know which sign s has at the roots of t, and conversely;

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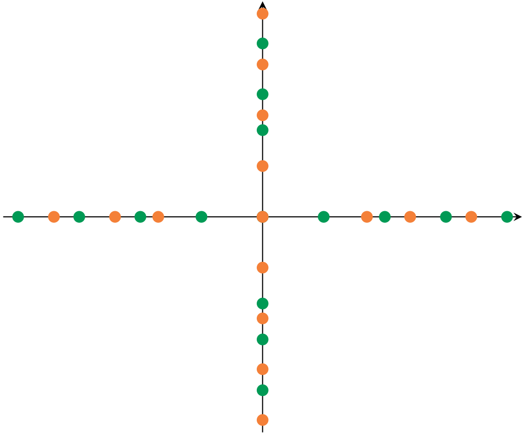
- Using Beatty's Theorem, we can *prove* that the roots of s and t are intertwined as we saw. This means that we know which sign s has at the roots of t, and conversely;
- We can use the intermediate value theorem on many intervals of the real and imaginary axes;
- We use the holomorphic implicit function Theorem;
- Since roots have multiplicity one, the symmetries of the problem imply that they stay on the axes!

Roots of $\lambda \mapsto F_1(\lambda)$

The continuation argument for $\tilde{F}_\gamma$ in the λ plane



Green: roots of s

Orange: roots of t

Roots of $\lambda \mapsto F_1(\lambda)$

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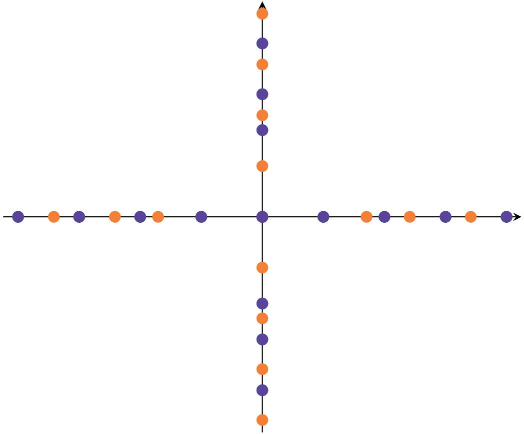


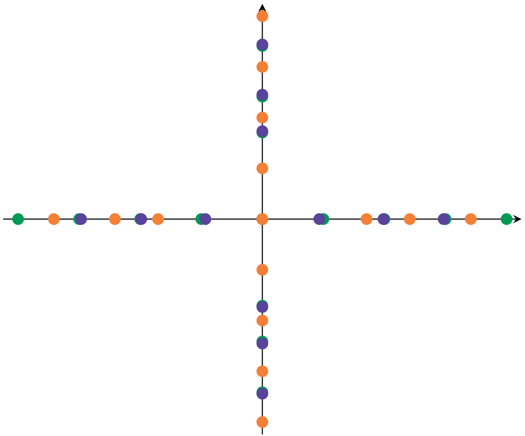
Green: roots of s

Orange: roots of t

$\gamma = 0$

Roots of $\lambda \mapsto F_1(\lambda)$

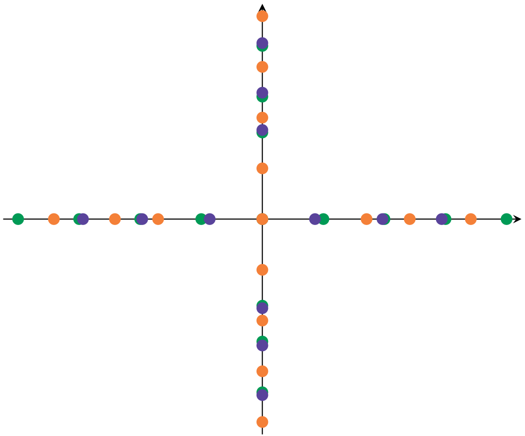
The continuation argument for $\tilde{F}_\gamma$ in the λ plane



Green: roots of s
Orange: roots of t
$\gamma = 0.1$

Roots of $\lambda \mapsto F_1(\lambda)$

The continuation argument for $\tilde{F}_\gamma$ in the λ plane



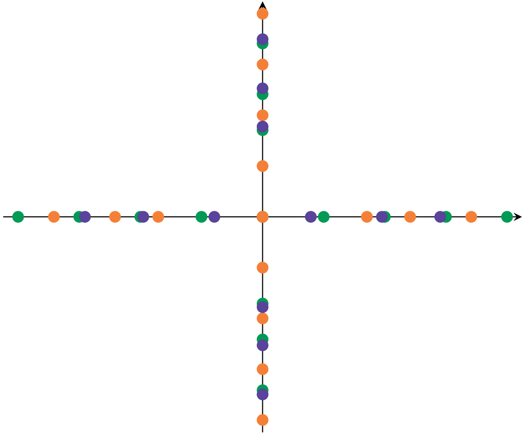
Green: roots of s

Orange: roots of t

$\gamma = 0.2$

Roots of $\lambda \mapsto F_1(\lambda)$

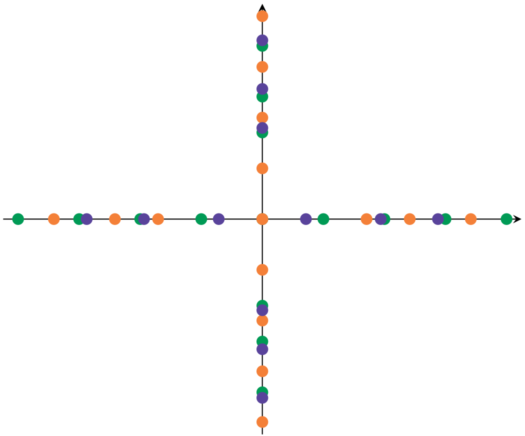
The continuation argument for $\tilde{F}_\gamma$ in the λ plane



Green: roots of s
Orange: roots of t
$\gamma = 0.3$

Roots of $\lambda \mapsto F_1(\lambda)$

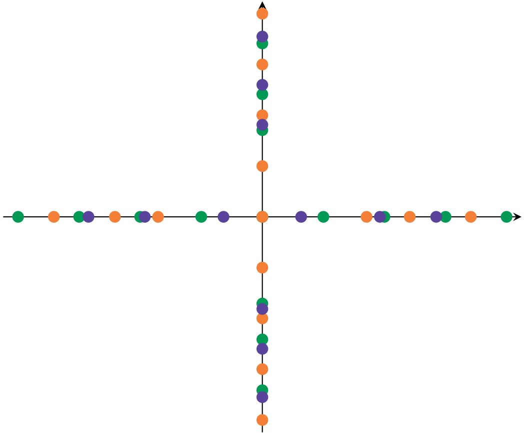
The continuation argument for $\tilde{F}_\gamma$ in the λ plane



Green: roots of s
Orange: roots of t
$\gamma = 0.4$

Roots of $\lambda \mapsto F_1(\lambda)$

The continuation argument for $\tilde{F}_\gamma$ in the λ plane



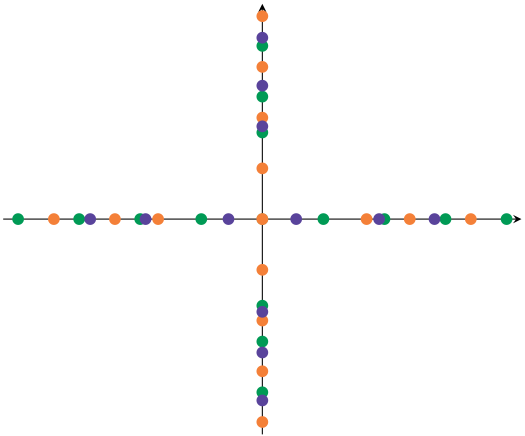
Green: roots of s

Orange: roots of t

$\gamma = 0.5$

Roots of $\lambda \mapsto F_1(\lambda)$

The continuation argument for $\tilde{F}_\gamma$ in the λ plane



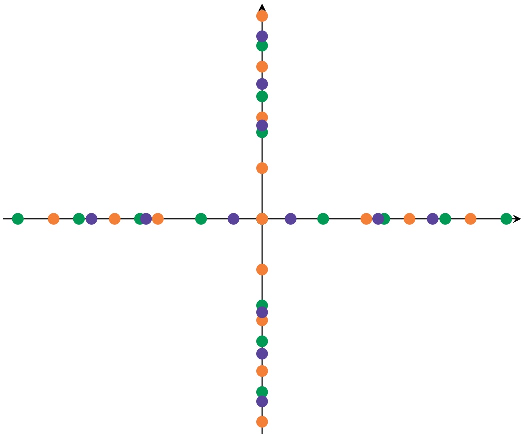
Green: roots of s

Orange: roots of t

$\gamma = 0.6$

Roots of $\lambda \mapsto F_1(\lambda)$

The continuation argument for $\tilde{F}_\gamma$ in the λ plane



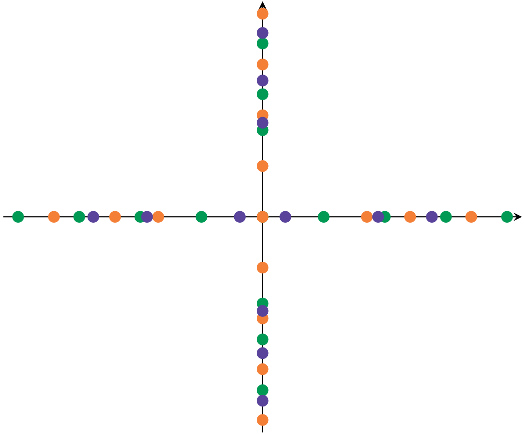
Green: roots of s

Orange: roots of t

$\gamma = 0.7$

Roots of $\lambda \mapsto F_1(\lambda)$

The continuation argument for $\tilde{F}_\gamma$ in the λ plane



Green: roots of s

Orange: roots of t

$\gamma = 0.8$

Roots of $\lambda \mapsto F_1(\lambda)$

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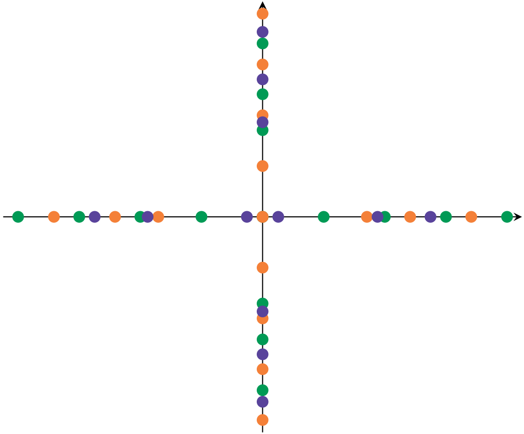


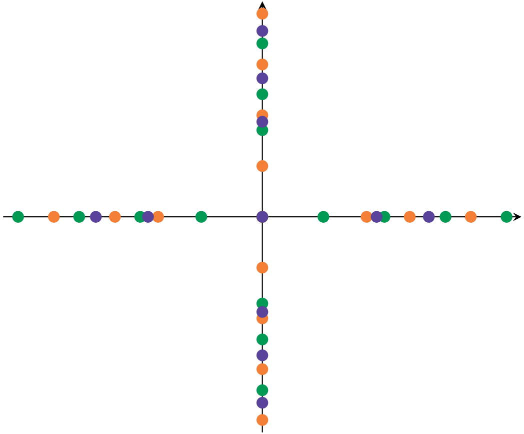
Green: roots of s

Orange: roots of t

$\gamma = 0.9$

Roots of $\lambda \mapsto F_1(\lambda)$

The continuation argument for $\tilde{F}_\gamma$ in the λ plane



Green: roots of s

Orange: roots of t

$\gamma = 1 \rightsquigarrow$ roots of F_1

Roots of $\mu \mapsto F_1(\lambda(\mu))$

Graphical representation in the μ plane

$$\lambda(\mu) = \sqrt[4]{-\frac{b\mu + \mu^2}{a}}$$

i.e.

$$\mu^2 + b\mu + a\lambda^4 = 0$$

Roots of $\mu \mapsto F_1(\lambda(\mu))$

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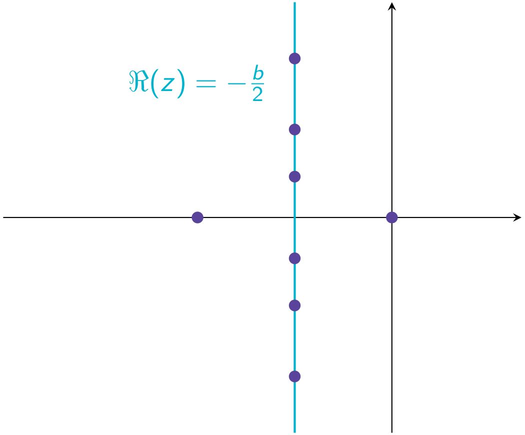

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The axis $\Re(z) = -\frac{b}{2}$

Lemma

If $\alpha > 0$, then, the only possible eigenvalue on the axis $\Re(z) = -\frac{b}{2}$ is $\mu = -\frac{b}{2}$.

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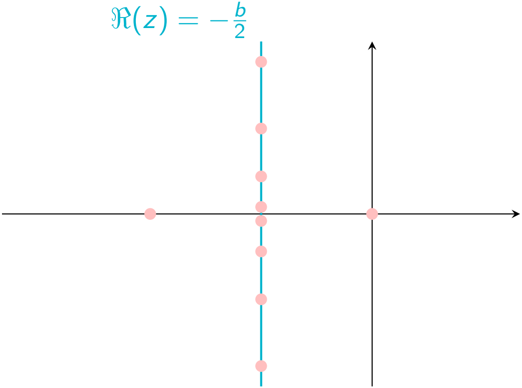
Lemma (Roughly stated)

When $\alpha > 0$ is small, "all eigenvalues μ starting on the axis $\Re(z) = -\frac{b}{2}$ move to the left".

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



Pink: roots of F_0 $(\alpha = 0)$

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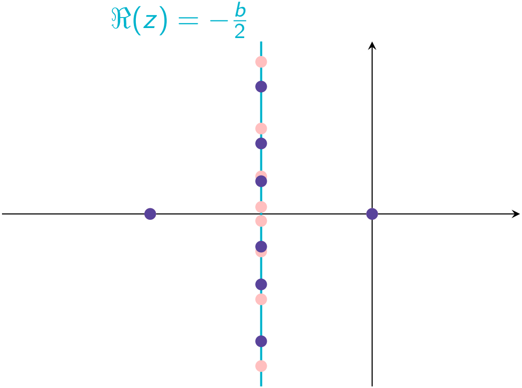

Pink: roots of F_0 $(\alpha = 0)$

Violet: roots of F_1 $(\alpha \to +\infty)$

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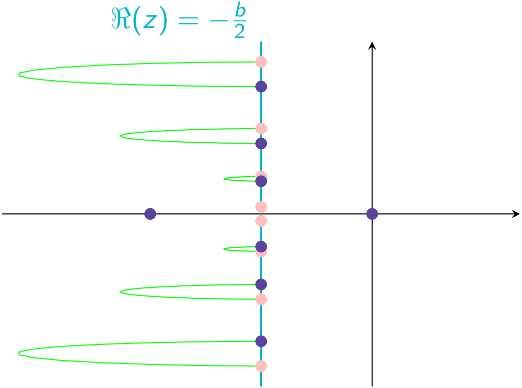

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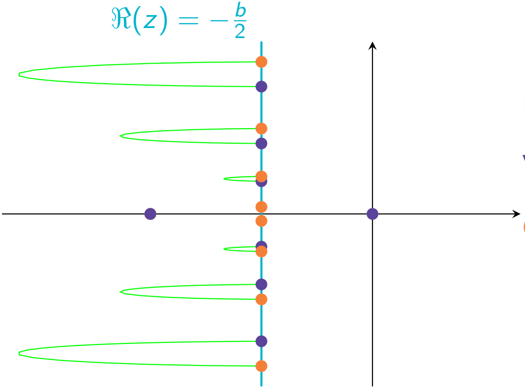


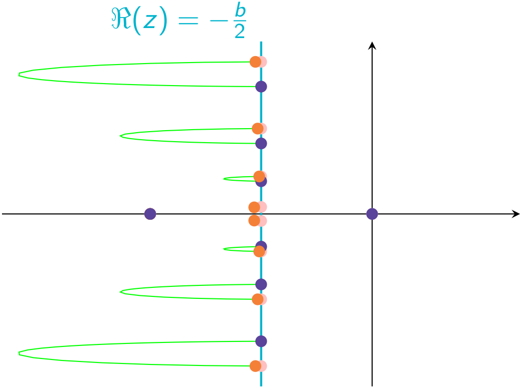
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 0$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



Pink: roots of F_0 $(\alpha = 0)$

Violet: roots of F_1 $(\alpha \to +\infty)$

Orange: eigenvalues $(\alpha = 0.1)$

Moving α from 0 to $+\infty$

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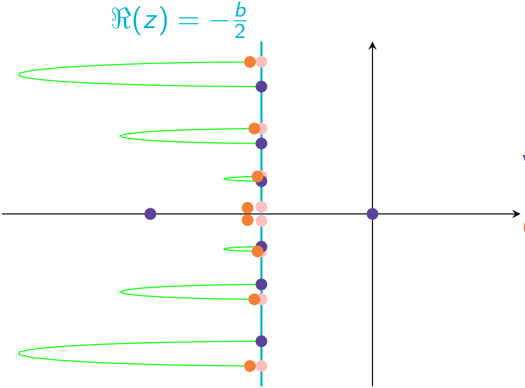


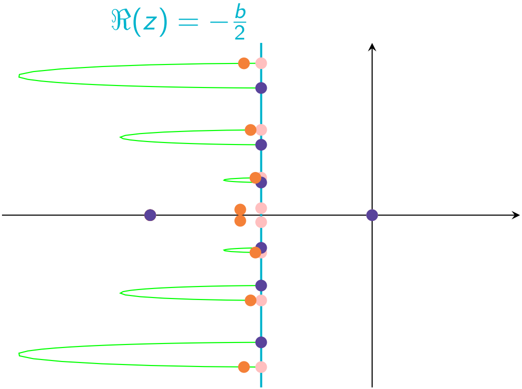
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 0.2$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 0.3$)

Moving α from 0 to $+\infty$

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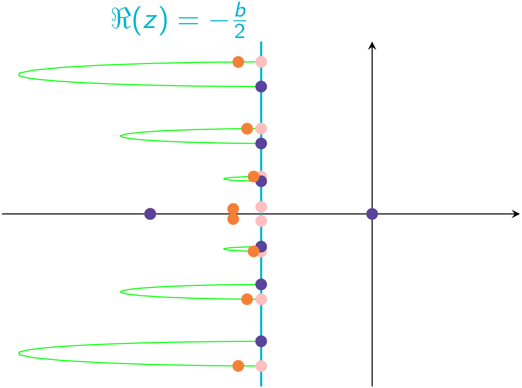


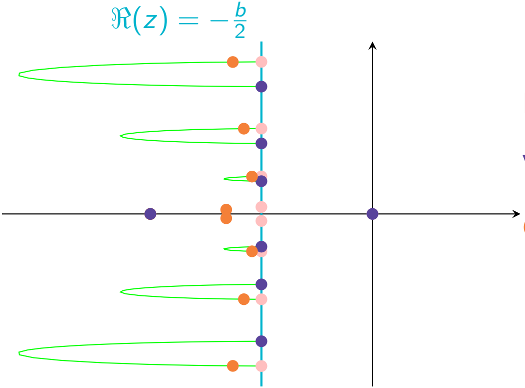
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 0.4$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



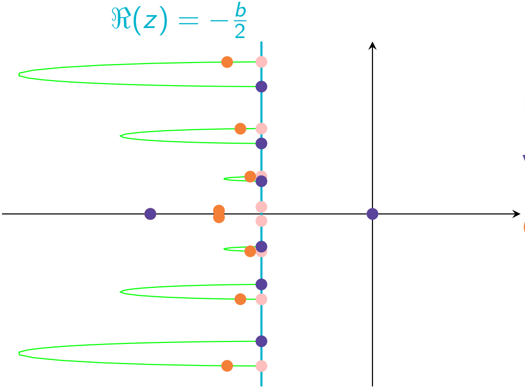
Pink: roots of F_0 $(\alpha = 0)$

Violet: roots of F_1 $(\alpha \to +\infty)$

Orange: eigenvalues $(\alpha = 0.5)$

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



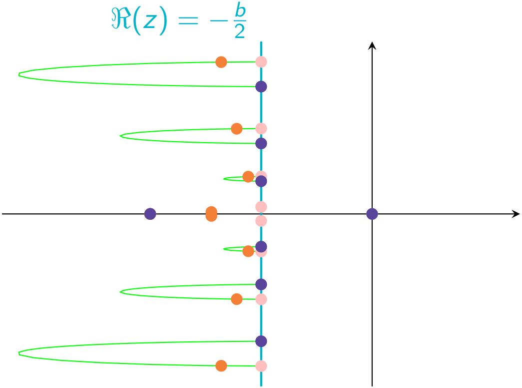
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 0.6$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



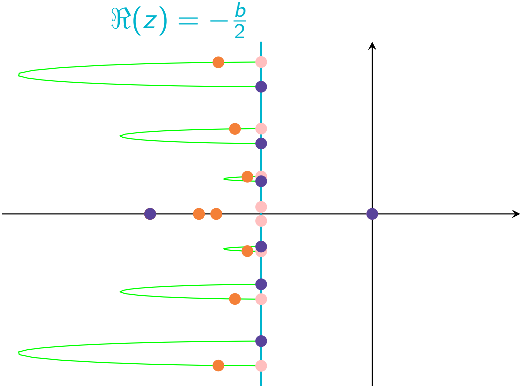
Pink: roots of F_0 $(\alpha = 0)$

Violet: roots of F_1 $(\alpha \to +\infty)$

Orange: eigenvalues $(\alpha = 0.7)$

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



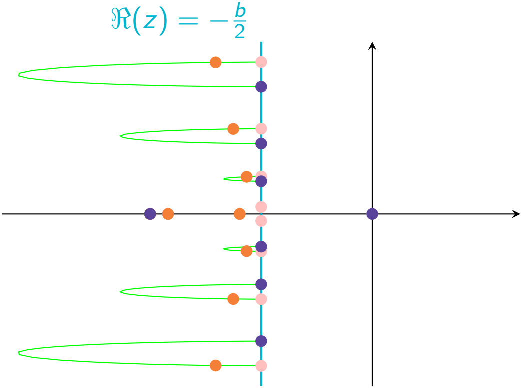
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 0.75$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



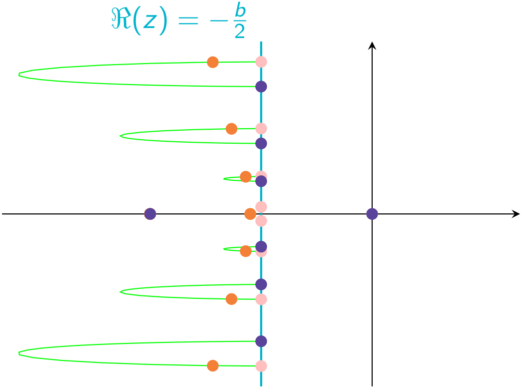
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 0.8$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



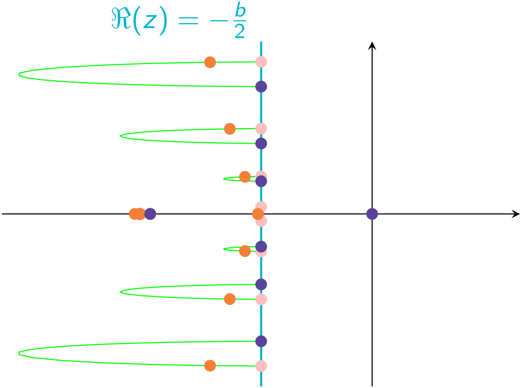
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 0.85$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



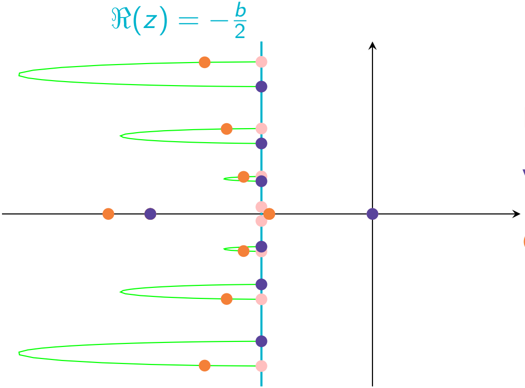
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 0.9$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



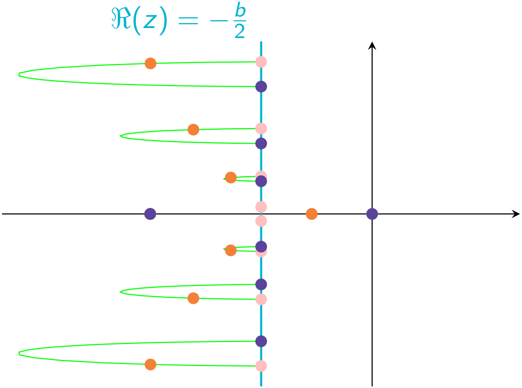
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 1$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



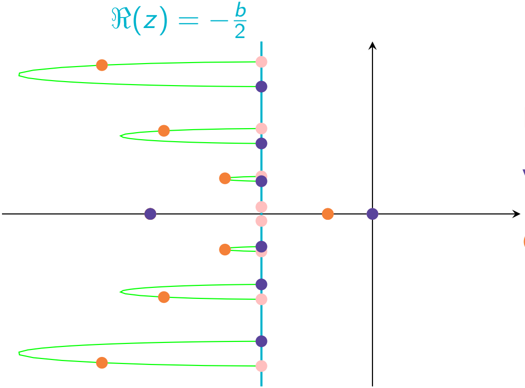
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 2$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



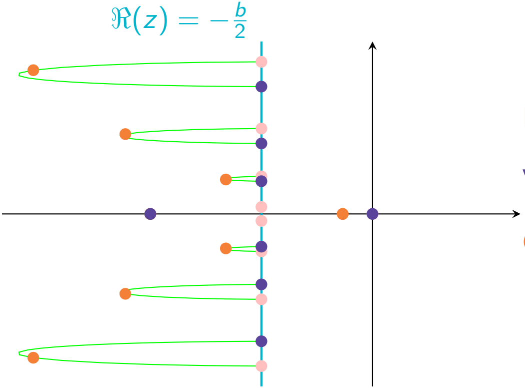
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 3$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



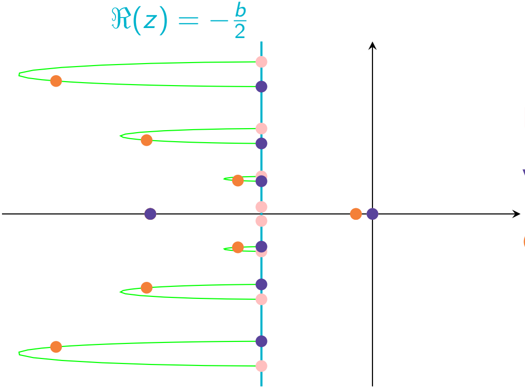
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 5$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



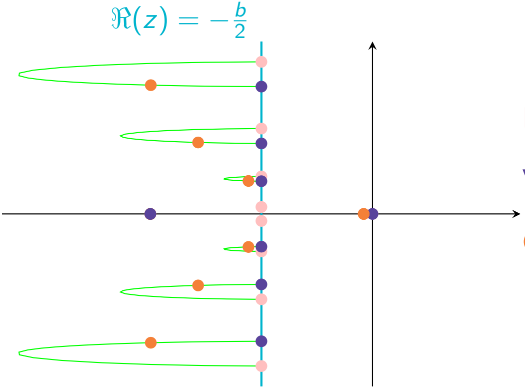
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 10$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



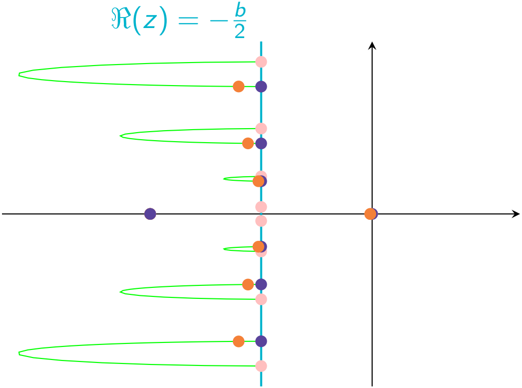
Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 20$)

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



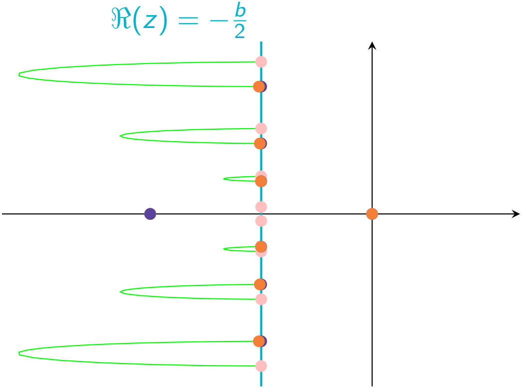
Pink: roots of F_0 $(\alpha = 0)$

Violet: roots of F_1 $(\alpha \to +\infty)$

Orange: eigenvalues $(\alpha = 100)$

Moving α from 0 to $+\infty$

Graphical representation in the μ plane: $a = 0.05, b = 3, \xi = \sqrt{2} - 1$



Pink: roots of F_0 ($\alpha = 0$)

Violet: roots of F_1 ($\alpha \to +\infty$)

Orange: eigenvalues ($\alpha = 1000$)

Moving α from 0 to $+\infty$

Main ingredients of the proofs

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- Applying the intermediate value Theorem many times between consecutive roots of F_0 and F_1.

A remarkable fact

Our methods will imply that the roots (other then a few known exceptions) have multiplicity one. When we know they are real, this implies that their derivative with respect to α **has a sign**, so that we obtain **monotonicity of real eigenvalues with respect to α!**

Main results

Theorem (G., Régnier, Troestler (2023))

Recall that the optimal decay rate $\omega(\alpha)$ is given by

$$\omega(\alpha) = \sup\Big\{\Re(\mu) \mid \mu \text{ is an eigenvalue of } \mathcal{A}_\alpha\Big\}.$$

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1. *ω is continuous in α.*
2. *ω is nondecreasing;*
3. *one has*

$$\lim_{\alpha\to+\infty} \omega(\alpha) = 0. \qquad (!)$$

A physical conclusion

The term $\alpha\partial_t u(\xi, t)\delta_\xi$ is **definitely not** a damping term in the problem.

Summary and perspectives

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A take-home question

Can we also study the optimal decay rate of an evolution problem involving variable coefficients $a(x)$ and $b(x)$?

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In this work, we made use of the *explicit* expression of the characteristic equation. This allows to obtain detailed information about the optimal decay rate, by using nonlinear analysis and complex analysis methods on the characteristic equation.

A take-home question

Can we also study the optimal decay rate of an evolution problem involving variable coefficients $a(x)$ and $b(x)$? Can one show that the damping rate converges to 0 for $\alpha \to +\infty$ in this case too?

Thanks for your attention!

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The axis $\Re(z) = -\frac{b}{2}$

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If $\alpha > 0$*, then, the only possible eigenvalue on the axis* $\Re(z) = -\frac{b}{2}$ *is* $\mu = -\frac{b}{2}$.

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$$\lambda^4 = -\frac{\mu(b+\mu)}{a} = \frac{|\mu|^2}{a}$$

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is positive. Plugging such a λ into the characteristic equation implies after some elementary computations that μ is real. □

Small values of α, $\alpha > 0$

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Ingredients of the proof:

- explicit expression of the derivative of eigenvalues with respect to α;
- knowledge of the signs of F_0 and F_1 (this required to work a lot in the previous slides!).